Xavier University of Louisiana
Math Fair Spring 2024

Math Competition (Grade 11-12)

Directions: The use of calculator is **not** permitted. Choose the answer for each of the following problem, circling the correct answer on this page and filling in the appropriate space on the answer sheet. Be careful and clear. **If none of the given choices is correct choose (E) on the answer sheet.**

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1. Find all solution of $2\cos^2(x) - \cos(x) - 1 = 0$ on the interval $[0, 2\pi]$.

   $(A)\ 0, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$ $(B)\ 0, 2\pi, \frac{\pi}{3}, \frac{2\pi}{3}$ $(C)\ 0, 2\pi, \frac{2\pi}{3}, \frac{4\pi}{3}$ $(D)\ \pi, 2\pi, \frac{2\pi}{3}, \frac{5\pi}{3}$

2. Suppose that b and c are constant and

   $$(x+2)(x+b) = x^2 + cx + 6.$$

   What is c?

   $(A)\ -5$ $(B)\ -3$ $(C)\ -1$ $(D)\ 3$ $(E)\ 5$

3. When you roll a single ordinary dice, which of the following is the most likely to be true about your score

   (A) it is odd
   (B) it is a factor of 18
   (C) it is prime
   (D) it is a factor of 12
   (E) it is even

4. If the equations $ax + 3y = 5$ and $2x + by = 3$ represent the same line in the xy-plane, then ab is equal to:

   $(A)\ -6$ $(B)\ -1$ $(C)\ 2$ $(D)\ 3$ $(E)\ 6$

5. What is the number of pairs (x, y) of real numbers satisfying

   $$|\tan(\pi y)| + \sin^2(\pi x) = 0 \text{ and } x^2 + y^2 \leq 2?$$

   $(A)\ 1$ $(B)\ 4$ $(C)\ 5$ $(D)\ 8$ $(E)\ 9$

6. Let P be a point on the cuve $y = \frac{1}{x} + 2$ whose x-coordinate is a number $r > 0$. Let A be the point $(2r, 0)$. Let B be the intersection of the line going through A and P with the y-axis. Find the area of the triangle AOB, where O is the origin.

   $(A)\ 2r(1+2r)$ $(B)\ 4r(1+2r)$ $(C)\ \frac{1}{2r} + 2$ $(D)\ 2(1+2r)$

7. Suppose that $P(x) = ax^4 + bx^2 + x + 5$ and that $P(-3) = 2$. What is $P(3)$?

   $(A)\ -5$ $(B)\ -2$ $(C)\ 1$ $(D)\ 3$ $(E)\ 8$

8. Suppose that $3 = k \cdot 2^r$ and that $15 = k \cdot 4^r$. What is r?

   $(A)\ -\log_2 5$ $(B)\ \log_5 2$ $(C)\ \log_{10} 5$ $(D)\ \log_2 5$ $(E)\ \frac{5}{2}$

9. For all real number x, except $x = 0$ and $x = 1$, the function f is defined by

$$f\left(\frac{x}{x-1}\right) = \frac{1}{x}.$$

Suppose $0 < \theta < \pi/2$. What is $f(\sec^2(\theta))$?

$(A)\ \sin^2(\theta)$ $(B)\ \cos^2(\theta)$ $(C)\ \tan^2(\theta)$ $(D)\ \cot^2(\theta)$ $(E)\ \csc^2(\theta)$

10. The equation $x^4 - 3x^3 - x^2 + 5x + 2 = 0$ has four solutions, a, b, c, and d. Calculate the value of $(1-a)(1-b)(1-c)(1-d)$.

$(A)\ 0$ $(B)\ 2$ $(C)\ 4$ $(D)\ 8$ $(E)\ 16$

11. If $f(x) = 2^x$, then 16^8 is equal to

$(A)\ f(7)$ $(B)\ f(12)$ $(C)\ f(f(5))$ $(D)\ f(f(3))$ $(E)\ f(f(f(f(3))))$

12. Suppose that

$$\log_2(\log_3(\log_5(\log_7 N))) = 11.$$

How many different prime numbers are factors of N?

$(A)\ 1$ $(B)\ 2$ $(C)\ 3$ $(D)\ 4$ $(E)\ 5$

13. Find the derivative of $f(x) = x\cos^3(2x)$

$(A)\ -3x\sin(2x)$
$(B)\ \cos^3(2x) + 6x\cos^2(2x)\sin(2x)$
$(C)\ \cos^3(2x) + 6x\cos^2(2x)$
$(D)\ \cos^3(2x) - 6x\cos^2(2x)\sin(2x)$

14. How many integers, x, are there so that

$$\left(x - \frac{1}{2}\right)\left(x - \frac{2}{3}\right)\left(x - \frac{3}{4}\right)\cdots\left(x - \frac{2023}{2024}\right) < 0?$$

$(A)\ 0$ $(B)\ 1$ $(C)\ 2023$ $(D)\ 2024$

15. Given a cube with edges of length 2. Let A and B be the midpoint of the two opposite sides of the base of the cube. What is the area of the triangle formed by three points A, B, and anyone of the corners of the top of the cube?

$(A)\ \frac{3\sqrt{2}}{2}$ $(B)\ \frac{3\sqrt{10}}{4}$ $(C)\ \sqrt{5}$ $(D)\ 3$ $(E)\ \frac{3\sqrt{5}}{2}$

16. Find a positive integer M so that the equation

$$(x - 12)(x + M) = M - 51$$

has exactly one solution.

$(A)\ 1$ $(B)\ 2$ $(C)\ 5$ $(D)\ 4$ $(E)\ 7$

17. Calculate

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

$(A)\ \frac{1}{5}$ $(B)\ \frac{8}{5}$ $(C)\ \frac{3}{5}$ $(D)\ \frac{4}{5}$ $(E)\ \frac{9}{5}$

18. If $x = \left(1 + \frac{1}{n}\right)^n$ and $y = \left(1 + \frac{1}{n}\right)^{n+1}$, express y^x as a power of x.

$(A)\ x^y \qquad (B)\ x^{1+y} \qquad (C)\ x^x \qquad (D)\ x^{y-1} \qquad (E)\ x^{2y}$

19. A rectangle has height two units less than its base. A square has side one unit less than the base of the rectangle. The rectangle has area A. What is the area of the square, in terms of A?

$(A)\ A \qquad (B)\ A+1 \qquad (C)\ A+2 \qquad (D)\ A-1 \qquad (E)\ A-2$

20. Find the exact value of $\sin(x)$ if $\tan(x) = \cos(x)$.

$(A)\ \frac{1+\sqrt{5}}{2} \qquad (B)\ \frac{1-\sqrt{5}}{2} \qquad (C)\ \frac{-1-\sqrt{5}}{2} \qquad (D)\ \frac{-1+\sqrt{5}}{2} \qquad (E)\ \frac{\sqrt{5}}{2}$

21. How many elements are there in the set

$$\{x\ :\ x \text{ a positive integer such that } -1 \le \log_{\frac{1}{x}} 10 < -\frac{1}{2}\}?$$

$(A)\ 10 \qquad (B)\ 30 \qquad (C)\ 50 \qquad (D)\ 70 \qquad (E)\ 90$

22. Find the value of

$$2024^2 - 2023^2 + 2022^2 - 2021^2 + \cdots + 4^2 - 3^2 + 2^2 - 1^2$$

$(A)\ 3049200 \qquad (B)\ 2049300 \qquad (C)\ 4029500 \qquad (D)\ 3029400 \qquad (E)\ 2039400$

23. Let ABC be a triangle with altitudes CD and AE, with $BD = 3$, $DA = 5$ and $BE = 2$. Find EC.

$(A)\ 6 \qquad (B)\ 8 \qquad (C)\ 10 \qquad (D)\ 13 \qquad (E)\ 15$

24. **Tie Breaker:** This is graded only to determine the first, second, or third place tie. Show your work.

A function $f(x)$ satisfies $f(1-x) = f(1+x)$ for every number x. If the equation $f(x) = 0$ has 8 real roots. What is the sum of all these 8 roots? Explain your answer.

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1. $2\cos^2(x) - \cos(x) - 1 = 0$

(C) $(2\cos x + 1)(\cos x - 1) = 0$

$\cos x = -\frac{1}{2} \qquad \cos x = 1$

$x = \frac{2\pi}{3}, \frac{4\pi}{3} \qquad x = 0, 2\pi$

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2. $(x+2)(x+b) = x^2 + cx + 6$

(E) $x^2 + bx + 2x + 2b = x^2 + cx + 6$

$b + 2 = c \qquad 2b = 6 \qquad b = 3 \qquad c = 5$

3. (A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{1}{2}$

(D)

(D) $\frac{5}{6}$ (E) $\frac{1}{2}$

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4. $ax + 3y = 5$

(E) $2x + by = 3$

$3ax + 9y = 15$

$10x + 5by = 15$

$10 = 3a \qquad 9 = 5b$

$90 = 15ab \qquad ab = 6$

$\frac{a}{3} = \frac{2}{b} \qquad ab = 6$

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5. $\tan(\pi y) = 0 \quad , \quad \sin \pi x = 0$

(E)

$\pi y = n\pi \qquad \pi x = m\pi$

$y = n \qquad x = m$

$(0,0)$, $(0,1)$, $(1,0)$, $(0,-1)$, $(-1,0)$

$(1,1)$ $(1,-1)$ $(-1,1)$ $(-1,-1)$

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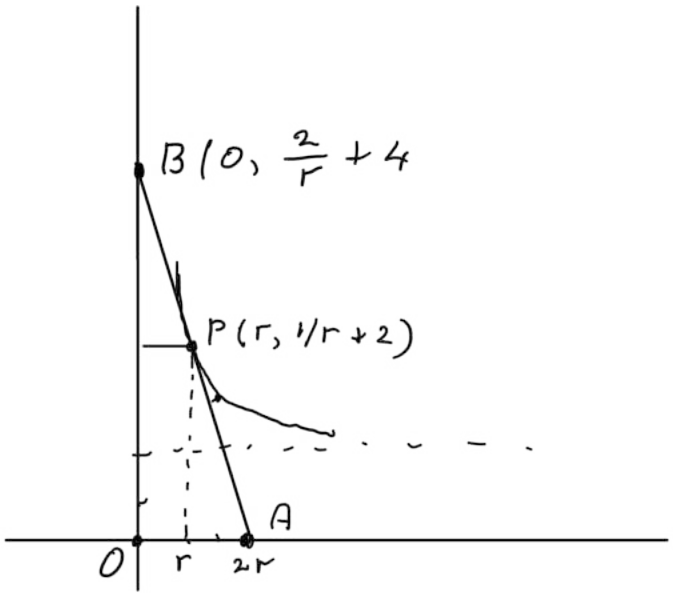
6. $y = \frac{1}{x} + 2$

(D)



$$\left(\frac{1}{r}+2\right)r + r\left(\frac{1}{r}+2\right)$$

$$= 1 + 2r + 1 + 2r$$

$$= 2 + 4r = 2(1+2r)$$

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7. (E)

$$P(x) = ax^4 + bx^2 + x + 5$$

$$P(-3) = 81a + 9b - 3 + 5 = 2$$

$$81a + 9b + 2 = 2$$

$$81a + 9b = 0$$

$$P(3) = 81a + 9b + 3 + 5 = 8$$

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8. (D)

$$3 = k2^r \qquad 15 = k4^r$$

$$15 = 3 \cdot 2^{-r} \cdot 2^{2r}$$

$$15 = 3 \cdot 2^r \qquad 5 = 2^r \qquad \log_2 5 = r$$

9. (A) $f\left(\frac{x}{x-1}\right) = \frac{1}{x}$

$y = \frac{x}{x-1}$ $\quad (x-1)y = x$

$xy - y = x$

$x(y-1) = y$ $\quad x = \frac{y}{y-1}$

$f(y) = \frac{y-1}{y} = 1 - \frac{1}{y}$

$f(\sec^2\theta) = 1 - \cos^2\theta = \sin^2\theta$

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10. (C) $x^4 - 3x^3 - x^2 + 5x + 2$

$= (x-a)(x-b)(x-c)(x-d)$

$1 - 3 - 1 + 5 + 2 = (1-a)(1-b)(1-c)(1-d)$

11

4

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11. (C) $f(x) = 2^x$

$16^8 = (2^4)^8 = 2^{32} = f(32)$

$f(f(5)) = 2^{f(5)} = 2^{32}$

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12. (A) $\log_2 \log_3 \log_5 \log_7 (N) = 11$

$$\log_3 \log_5 \log_7 (N) = 2^{11}$$

$$\log_5 \log_7 (N) = 3^{2^{11}}$$

$$\log_7 (N) = 5^{3^{2^{11}}}$$

$$N = 7^{5^{3^{2^{11}}}}$$

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13. (D)

$$f(x) = x\cos^3(2x)$$

$$f'(x) = \cos^3(2x) - 6x\cos^2(2x)\sin(2x)$$

14.

15. (C)

$A = (-1, 0, 0) \qquad B = (1, 0, 0)$

$C = (1, 1, 2)$

$\overrightarrow{AC} = (2, 1, 2) \qquad \overrightarrow{AB} = (2, 0, 0)$

$$\begin{pmatrix} i & j & k \\ 2 & 1 & 2 \\ 2 & 0 & 0 \end{pmatrix} = 2(2j - k)$$

$$= 2(0, 2, -1)$$

$$\text{Area} = \frac{1}{2} \cdot 2 \, \|(0, 2, -1)\| = \sqrt{4 + 1} = \sqrt{5}$$

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16. (B)

$$(x - 12)(x + M) = M - 51$$

$$x^2 + Mx - 12x - 12M - M + 51 = 0$$

$$x^2 + (M - 12)x - 13M + 51 = 0$$

$$(M - 12)^2 - 4(51 - 13M) = 0$$

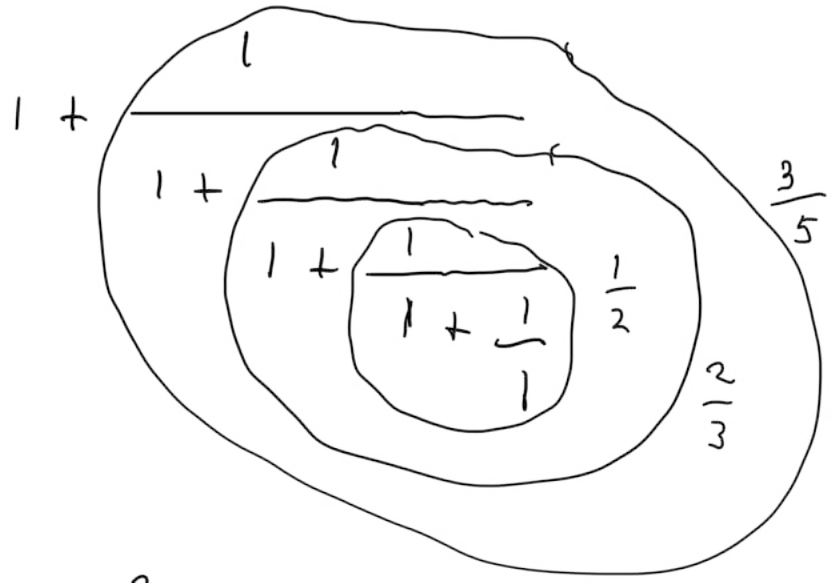
$$M^2 - 24M + 144 - 204 + 52M = 0$$

$$M^2 + 28M - 60 = 0$$

$$\frac{-28 \pm \sqrt{784 + 240}}{2} = \frac{-28 + 32}{2} = 2$$

17.

(B)



$\frac{8}{5}$

18.

(A)

$$x = \left(1 + \frac{1}{n}\right)^n \qquad y = \left(1 + \frac{1}{n}\right)^{n+1}$$

$$y^x = \left[\left(1 + \frac{1}{n}\right)^{n+1}\right]^{\left(1 + \frac{1}{n}\right)^n}$$

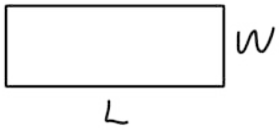
$$= \left(1 + \frac{1}{n}\right)^{(n+1)\left(1 + \frac{1}{n}\right)^n}$$

$$= \left(1 + \frac{1}{n}\right)^{n\left(1 + \frac{1}{n}\right)^{n+1}}$$

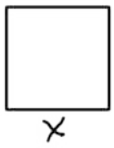
$$= x^y$$

19.

(B)



$W = L - 2$



$x = L - 1$

$LW = A \qquad x^2 = (L-1)^2 = L^2 - 2L + 1$

$A = L(L-2) = L^2 - 2L$

$x^2 = A + 1$

20.

(D)

$$\tan x = \cos x$$

$$\sin x = \cos^2 x$$

$$\sin x = 1 - \sin^2 x$$

$$\sin^2 x + \sin x - 1 = 0$$

$$\sin x = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$= \frac{\sqrt{5} - 1}{2}$$

21.

(E)

$$-1 \leq \log_{\frac{1}{x}} 10 < -\frac{1}{2}$$

$$\left(\frac{1}{x}\right)^{-1} \geq 10 > \left(\frac{1}{x}\right)^{-1/2}$$

$\sqrt{x} < 10 \leq x$

$10, 11, \cdots, 99$

$99 - 9 = 90$

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22. (B)

$2024^2 - 2023^2 = 2024 + 2023$

$2022^2 - 2021^2 = 2022 + 2021$

$\vdots$

$+ \cdots + 2 + 1$

$$1 + 2 + \cdots + 2024 = \frac{2024(2025)}{2}$$

$$= 2049300$$

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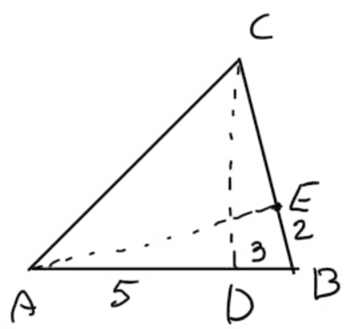
23. (C)



$$\frac{\overline{CD} \cdot 8}{2} = \frac{\overline{BC} \cdot \overline{AE}}{2}$$

$$8\overline{CD} = (\overline{EC} + 2) \cdot \overline{AE}$$

$$\overline{AE} = \sqrt{8^2 - 4} = \sqrt{60}$$

$8\overline{CD} = (\overline{EC}+2)\sqrt{60}$ $\quad \overline{CD} = (\overline{EC}+2)\frac{\sqrt{60}}{8}$

$\overline{CD}^2 + 9 = (EC+2)^2$

$(EC+2)^2\frac{60}{64} + 9 = (EC+2)^2$

$(EC+2)^2 = x$

$\frac{60}{64}x + 9 = x$

$9 = \left(1 - \frac{60}{64}\right)x$

$9 = \frac{4}{64}x = \frac{1}{16}x$

$x = 144 = (\overline{EC}+2)^2$

$EC + 2 = 12$

$EC = 10$

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24. $f(1-x) = f(1+x)$

$g(x) = f(x+1)$

$g(-x) = f(1-x) = f(1+x) = g(x)$

$f(x_i) = 0 \quad g(x_i - 1) = 0$

$x_i - 1$; $i = 1, \ldots, 8$ are zeros of g

$\Sigma(x_i - 1) = 0 \quad \Sigma x_i - 8 = 0$ $\quad \boxed{\Sigma x_i = 8}$